

Complete the problems on the next five pages as a 'warm up' before attempting the six practice achievement standards in our PEA 3 Differentiation Booklet.

QUESTION ONE

Differentiate the following.

(a) $y = 2\sqrt{x} - \frac{2}{x}$

(b) $y = 2(3x + 1)^3$

(c) $y = \sqrt{1 - x^2}$

(d) $y = -4e^{2x-1}$

(e) $y = 5 \ln(4x + 3)$

(f) $y = 2 \cos 4x$

(g) $y = 6 \sec\left(\frac{x}{3}\right)$

(h) $y = e^{\tan x}$

(i) $y = 3x^2 \ln x$

(j) $y = \frac{3x+1}{2x-3}$

QUESTION TWO

- (a) Find the gradient of the tangent to the curve $y = \frac{2}{x^2-1}$ at the point (3, 0.25).

- (b) Find the gradient of the normal to the curve $y = 4\sqrt{x} - x$ at $x = 9$.

QUESTION THREE

- (a) Find the equation of the tangent to the curve $f(x) = (2x + 1)^4$ at the point $(-1, 1)$.

- (b) Find the equation of the normal to the curve $f(x) = 2 \sin \frac{1}{2}x$ at $x = \frac{2\pi}{3}$.

QUESTION FOUR

- (a) Find the coordinates of the stationary points of the curve $y = x + \frac{1}{x}$.

- (b) Find the coordinates of the stationary points of the curve $y = \frac{4x}{x^2 + 1}$.

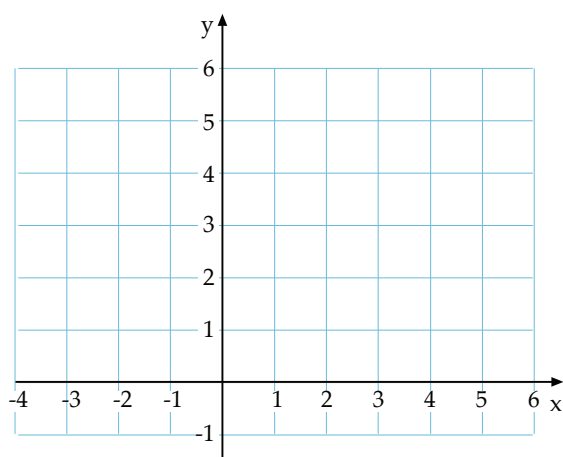
QUESTION FIVE

- (a) Find the x coordinate(s) of the point of inflection of the graph $f(x) = 9\ln(x+3) + \frac{x^2}{8}$.

- (b) When is the curve $f(x) = 2x^2 - \ln 2x$ increasing?

QUESTION SIX

- (c) On the axis below, sketch a graph of $f(x)$ that:
- is continuous for $-3 < x < 0$ and $0 < x < 4$ and is discontinuous when $x = 0$.
 - $f''(x) > 0$ for $-3 < x < 0$.
 - has $f'(x) = 0$ at $(-1, 1)$.
 - has $\lim_{x \rightarrow 0} f(x) = 2$, but $f(0) = 3$.
 - is not differentiable at $(1, 4)$.
 - has $f'(x) < 0$ for $x > 1$.



QUESTION SEVEN

- (a) If $x = \frac{2}{t}$ and $y = 4t^2$, find the gradient of the normal at $t = 1$.

- (b) If $x = 4 \cos \theta$ and $y = 2 \tan \theta$, find the gradient of the tangent at $\theta = \frac{\pi}{4}$.

- (c) If $x = \sin^2 \theta$ and $y = 2 \cos \theta$, find the gradient of the normal at $\theta = \frac{\pi}{3}$.

QUESTION EIGHT

- (a) The height of a kite as it soars into the air can be modelled by the function

$$H = \left(\frac{t}{6} + 1\right)^3$$

where H is the height in metres and t is the time in seconds.

What is the rate of change in the height of the kite after 8 seconds?

- (b) Coffee is poured into a cup and allowed to cool to room temperature. The temperature T of the liquid after t minutes is modelled by the formula

$$T = 70e^{-0.3t} + 25.$$

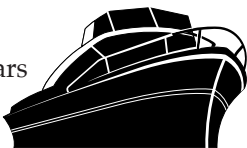
What is the rate of change of the temperature after 4 minutes?

QUESTION NINE

- (a) The cost of running a boat on a trip is given by

$$C = 4.5v^2 + \frac{1200}{v}, \quad v \neq 0$$

where C is the cost in dollars and v is the average speed in km/hr.



Find the average speed for which the cost is a minimum and find the minimum cost. (You may assume $\frac{d^2C}{dv^2} > 0$).

- (b) A museum is holding a special exhibition. It records the number of visitors over several weeks. The number of people who attend can be modelled by the equation $P = 6000te^{-t/3}$, where t is the time in weeks.

In which week does the maximum number of visitors attend the exhibition and how many people attend?

QUESTION TEN

A spherical balloon is inflated at the rate of 1.15 m^3 per minute.

What is the volume of the balloon when the radius is increasing at 0.15 m per minute?

QUESTION TWELVE

The sum of two numbers, x and y , is a fixed number, S .

Use differentiation to show that the product is a maximum when $x = \frac{2S}{3}$.

Answers – Readiness Check

Question One:

- $\frac{dy}{dx} = \frac{1}{\sqrt{x}} + \frac{2}{x^2}$
- $\frac{dy}{dx} = 6(3x+1)^2 \cdot 3 = 18(3x+1)^2$
- $\frac{dy}{dx} = 0.5(1-x^2)^{-1/2} \cdot -2x = \frac{-x}{\sqrt{1-x^2}}$
- $\frac{dy}{dx} = -4e^{2x-1} \cdot 2 = -8e^{2x-1}$
- $\frac{dy}{dx} = \frac{5}{4x+3} \cdot 4 = \frac{20}{4x+3}$
- $\frac{dy}{dx} = -2 \sin 4x \cdot 4 = -8 \sin 4x$
- $\frac{dy}{dx} = 2 \sec\left(\frac{x}{3}\right) \tan\left(\frac{x}{3}\right)$
- $\frac{dy}{dx} = e^{\tan x} \cdot \sec^2 x$
- $\frac{dy}{dx} = 3x^2 \cdot \frac{1}{x} + 6x \ln x = 3x + 6x \ln x$
- $\frac{dy}{dx} = \frac{3(2x-3) - 2(3x+1)}{(2x-3)^2} = \frac{-11}{(2x-3)^2}$

Question Three: (a) $f'(x) = 4(2x+1)^3 \cdot 2 = 8(2x+1)^3$

At $x = -1$, $f'(x) = -8$

Equation: $y - 1 = -8(x + 1)$
 $y = -8x - 7$

(b) $f'(x) = \cos 0.5x$ At $x = \frac{2\pi}{3}$, $f'(x) = 0.5$

so perpendicular gradient = -2.

At $x = \frac{2\pi}{3}$, $f(x) = 1.732$ or $\sqrt{3}$.

Eqn: $y - 1.732 = -2\left(x - \frac{2\pi}{3}\right)$
 $y = -2x + 5.92$

Question Five: (a) $f'(x) = \frac{9}{x+3} + \frac{x}{4}$
 $f''(x) = \frac{-9}{(x+3)^2} + \frac{1}{4}$

At a point of inflection, $f''(x) = 0$

$\frac{-9}{(x+3)^2} + \frac{1}{4} = 0$

$\frac{-9}{(x+3)^2} = \frac{-1}{4}$

$-(x+3)^2 = -36$

$x = 3$ is the only solution.

(b) $f'(x) = 4x - \frac{1}{x}$
 $4x - \frac{1}{x} = 0$
 $4x^2 = 1$
 $x^2 = \frac{1}{4}$
 $x = \frac{1}{2}$ so increasing when $x > \frac{1}{2}$.

Question Two: (a) $\frac{dy}{dx} = -2(x^2-1)^{-2} \cdot 2x$
 $= \frac{-4x}{(x^2-1)^2}$

At $x = 3$, $\frac{dy}{dx} = \frac{-3}{16}$

(b) $\frac{dy}{dx} = \frac{2}{\sqrt{x}} - 1$

At $x = 9$, $\frac{dy}{dx} = \frac{-1}{3}$

Normal has gradient = 3

Question Four: (a) $\frac{dy}{dx} = 1 - \frac{1}{x^2}$

$1 - \frac{1}{x^2} = 0$

$x^2 = 1$

$x = \pm 1$

Max at $(-1, -2)$

and min at $(1, 2)$

(b) Using quotient rule

$\frac{dy}{dx} = \frac{4(x^2+1) - 8x^2}{(x^2+1)^2}$

and $\frac{4(x^2+1) - 8x^2}{(x^2+1)^2} = 0$

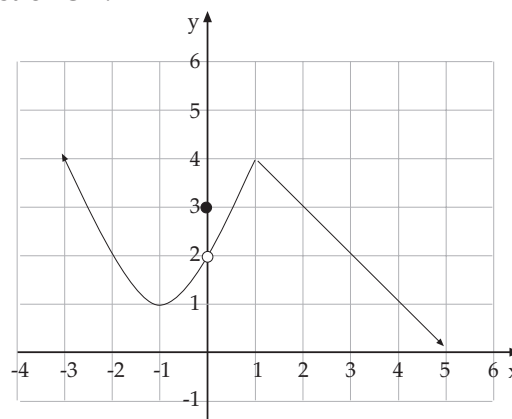
$-4x^2 + 4 = 0$

$x = 1, -1$

Coordinates are $(1, 2)$

and $(-1, -2)$

Question Six:



Answers – Readiness Check

Question Seven: (a) $\frac{dx}{dt} = \frac{-2}{t^2}, \frac{dy}{dt} = 8t$
 $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -4t^3$
 At $t = 1, \frac{dy}{dx} = -4$
 Normal has gradient $\frac{1}{4}$.

(b) $\frac{dx}{d\theta} = -4\sin\theta, \frac{dy}{d\theta} = 2\sec^2\theta$
 $\frac{dy}{dx} = \frac{2\sec^2\theta}{-4\sin\theta} = \frac{-2}{\cos^2\theta} \div 4\sin\theta$
 At $\theta = \frac{\pi}{4}, \frac{dy}{dx} = -\sqrt{2}$

(c) $\frac{dx}{d\theta} = 2\sin\theta\cos\theta$
 $\frac{dy}{d\theta} = -2\sin\theta$
 $\frac{dy}{dx} = \frac{-2\sin\theta}{2\sin\theta\cos\theta} = \frac{-1}{\cos\theta}$
 At $\theta = \frac{\pi}{3}, \frac{dy}{dx} = -2$
 Gradient of normal = $\frac{1}{2}$

Question Ten:

$$\frac{dV}{dt} = 1.15 \text{ m}^3/\text{min}$$

$$V = \frac{4}{3}\pi r^3 \text{ and } \frac{dV}{dr} = 4\pi r^2$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{4\pi r^2} \times 1.15 = \frac{1.15}{4\pi r^2}$$

$$\frac{1.15}{4\pi r^2} = 0.15 \text{ and } r = 0.781 \text{ m}$$

$$\text{Volume} = \frac{4}{3} \times \pi \times 0.781^3$$

$$= 1.995 \text{ m}^3$$

Question Twelve:

Product $P = x\sqrt{y}$
 $x + y = S \Rightarrow x = S - y$
 $P = (S - y)y^{1/2}$
 $P = Sy^{1/2} - y^{3/2}$
 $\frac{dP}{dy} = \frac{1}{2}Sy^{-1/2} - \frac{3}{2}y^{1/2}$
 $\frac{1}{2}S - \frac{3}{2}y = 0$
 $S = 3y \Rightarrow y = \frac{S}{3} \text{ and } x = \frac{2S}{3}$

Question Eight: (a) $\frac{dH}{dt} = 3\left(\frac{t}{6} + 1\right)^2 \cdot \frac{1}{6}$
 $= \frac{1}{2}\left(\frac{t}{6} + 1\right)^2$
 At $t = 8, \frac{dH}{dt} = 2.72 \text{ m/s}$

(b) $\frac{dT}{dt} = 70e^{-0.3t} \cdot -0.3 = -21e^{-0.3t}$
 At $t = 4, \frac{dT}{dt} = -6.325^\circ$
 Cooling at 6.3° per minute.

Question Nine: (a) $\frac{dC}{dv} = 9v - \frac{1200}{v^2}$
 $9v - \frac{1200}{v^2} = 0$
 $9v^3 = 1200$
 $v = \sqrt[3]{\frac{1200}{9}}$
 $v = 5.1 \text{ km/hr}$
 Minimum cost = \$352.34

(b) $\frac{dP}{dt} = 6000e^{\frac{-t}{3}} - 2000te^{\frac{-t}{3}}$ (prod. rule)
 $e^{\frac{-t}{3}}(6000 - 2000t) = 0$
 $t = 3$
 Maximum number in 3rd week.
 People = 6621 or 6622

Question Eleven:

Area: $2000 = 2rh - \frac{\pi r^2}{2}$
 so $h = \frac{1000}{r} + \frac{\pi r}{4}$
 Perimeter: $P = 2r + 2h + \pi r$
 $P = 2r + 2\left(\frac{1000}{r} + \frac{\pi r}{4}\right) + \pi r$
 $P = 2r + \frac{2000}{r} + \frac{3\pi r}{2}$
 $\frac{dP}{dr} = 2 - \frac{2000}{r^2} + \frac{3\pi}{2}$
 Setting $\frac{dP}{dr} = 0$ and solving
 $r = 17.26 \text{ cm and } h = 71.5 \text{ cm}$
 Minimum perimeter = 231.7 cm